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Existence of conditional shape invariance symmetry for singular power potential $V(r) = ar^2 + br^{-4} + cr^{-6}$

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Abstract
We examine the solvability conditions of the singular power potential, $V(r) = ar^2 + br^{-4} + cr^{-6}$, $a > 0$, $c > 0$, in the light of shape invariance. We prove that one energy level can be obtained algebraically if a corresponding constraint condition is satisfied by the parameters. We show that energies of, at most, two levels can be obtained algebraically, for a severely restricted set of parameters resulting from the satisfaction of two such constraint conditions, leading to the existence of conditional shape invariance.

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A central singular power potential of the form

$$V(r) = ar^2 + br^{-4} + cr^{-6} \qquad a > 0,\ c > 0 \tag{1}$$

has attracted considerable attention, due to the fact that conditionally exact solutions (CES) of the potential are possible. Znojil first investigated bound state solutions in this potential for $c = 0$ [1] and for $c \neq 0$ [2]. The potential (1) was also discussed in a different context in [3]. Starting with an ansatz, for the wavefunction, Kaushal [4] first obtained an analytic solution for the ground state, provided a certain constraint is satisfied by the potential parameters. Later Kaushal and Parashar [5] extended the method for the first excited state. However, Landtman [6] performed an accurate numerical calculation and showed that for the parameters chosen by Kaushal and Parashar [5], although the ground state energy that they obtained in [5] agrees with the numerical calculation, their first excited state energy does not. In an effort to resolve the fallacy, Varshni [7] obtained four sets of solutions, including one constraint equation for each set and showed that the analytic expression for the energy agrees with the numerical result for any one among the ground, the first and the second excited states, depending on

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the particular constraint condition satisfied. The agreement between analytic and numerical results is for only one state for a given set of parameter values.

In the above papers, the authors obtained the solutions from different mathematical points of view, however no attempt was made to throw any light on the underlying symmetry which is the reason behind the existence of CES of only one or, at most, two states in this potential. Since supersymmetric quantum mechanics (SSQM) in association with the shape invariance condition [8] is the most compelling technique to understand the analytical solvability of a potential, in this paper we critically examine this problem in the light of SSQM to understand the peculiarity from the point of view of the shape invariance condition and we point out that conditional shape invariance is the underlying symmetry of this potential. We further show that the energies of, at most, two levels can be obtained algebraically, only if the parameters are restricted severely. It is well known [8] that a shape invariant potential admits exact algebraic solutions in closed form for the entire spectrum. The fact that the potential in equation (1) has an exact solution, albeit under a constraint condition on its parameters, raises the question of its shape invariance. In this paper, we see that the potential, given by equation (1), has the basic structure of a shape invariant potential, although it requires a new constraint condition for each passage from one member to the next one in the hierarchy of partner potentials. Thus, the shape invariance is satisfied in a single step (when the corresponding constraint is satisfied), in the process of construction of the hierarchy of partner potentials.

Since each step has a different constraint condition, the satisfaction of this guarantees shape invariance between only two partners among the hierarchy of partner potentials. Consequently, the energy of a single level (either the ground state or any excited state) can be obtained algebraically. We see in the following that the constraint conditions of, at most, two separate steps (in the process of construction of partner potentials) can have a non-trivial solution. Hence, energies of, at most, two levels can be obtained algebraically; however, in this case the parameters of $V(r)$ are severely restricted by the two constraint conditions.

For the analysis of the potential of equation (1) in the light of SSQM, we need the superpotential $[W(r)]$, for which we start with an ansatz

$$W(r) = -\frac{A}{r} - Br + \frac{D}{r^3}, \qquad B < 0,\ D < 0, \tag{2}$$

where A, B and D are constants to be determined later. Acceptable behaviour of $W(r)$ for the limits of $r \to 0$ and ∞ demand that

$$B < 0, \qquad D < 0. \tag{3}$$

We choose units such that $\hbar/\sqrt{2m} = 1$. Let E_0 be the ground state energy of the lth partial wave in $V(r)$, then the effective potential in the shifted energy scale (such that the ground state is at zero energy) is

$$\begin{aligned} V_1(r) &= V(r) + \frac{l(l+1)}{r^2} - E_0 \\ &= ar^2 + br^{-4} + cr^{-6} + \frac{l(l+1)}{r^2} - E_0. \end{aligned} \tag{4}$$

According to SSQM, $V_1(r)$ is given by [8]

$$V_1(r) = W^2 - W'. \tag{5}$$

From equations (2), (4) and (5), the unknowns A, B and D satisfy

$$A^2 - 2BD - A = l(l+1) \tag{6}$$

$$B^2 = a, \qquad D^2 = c \tag{7}$$

$$3D - 2AD = b \tag{8}$$

$$2AB + B = -E_0. \tag{9}$$

Equation (7), together with the condition (3) yield

$$B = -\sqrt{a}, \qquad D = -\sqrt{c}. \tag{10}$$

From equations (10) and (8), we get

$$A = \left(3 + \frac{b}{\sqrt{c}}\right)\Big/2. \tag{11}$$

With the values of A and B substituted in equation (9), we get the ground state energy as

$$E_0 = \sqrt{a}\left(4 + \frac{b}{\sqrt{c}}\right). \tag{12}$$

At this stage equation (6) remains to be satisfied and leads to a constraint condition

$$(2\sqrt{c} + b)^2 = 4cl(l+1) + c(1 + 8\sqrt{ac}). \tag{13}$$

Equations (12) and (13) are identical with equations (4) and (5) of [5]. This shows that the ground state energy is obtained algebraically from equation (12), provided the constraint condition (13) is satisfied by the parameters of the effective potential

$$V_{eff}(r) = V(r) + \frac{l(l+1)}{r^2}. \tag{14}$$

The ground state wavefunction up to a normalization constant is given by [8]

$$\begin{aligned}\Psi_0 &= N_0 \mathrm{e}^{-\int W(r)\,\mathrm{d}r} \\ &= N_0 r^{(3+(b/\sqrt{c}))/2} \mathrm{e}^{(-(\sqrt{a}/2)r^2 - (\sqrt{c}/2)r^{-2})}.\end{aligned} \tag{15}$$

To investigate, if the potential $V_{eff}(r)$ is shape invariant, we construct the partner potential $V_2(r)$, according to the SSQM prescription [8]

$$\begin{aligned}V_2(r) &= W^2 + W' \\ &= ar^2 + b'r^{-4} + cr^{-6} + \frac{l'(l'+1)}{r^2} + R,\end{aligned} \tag{16}$$

where the new parameters are given in terms of the original ones through

$$l'(l'+1) = l(l+1) + 3 + \frac{b}{\sqrt{c}} \tag{17}$$

$$b' = b + 6\sqrt{c} \tag{18}$$

$$R = -E_0 - 2B = -E_0 + 2\sqrt{a}. \tag{19}$$

The mathematical form of $V_2(r)$ (equation (16)) is the same as that of $V_1(r)$ (equation (4)). Hence it appears that the potential $V_1(r)$ is shape invariant, with translation of parameters $l'(l'+1)$ and b' given by equations (17) and (18) respectively. However this shape invariance is not unconditionally valid, since constraint condition (13) has to be satisfied. So the potential admits conditional shape invariance between first two partners, which is responsible for the conditional exact solution for the ground state in the potential. Even if the parameters of $V_{eff}(r)$ satisfy equation (13), shape invariance is not valid in successive steps. To see this we repeat our procedure with $V_2(r)$ as the starting potential, defining

$$\hat{V}(r) = V_2(r). \tag{20}$$

If $\hat{E}_0$ is the ground state energy in $\hat{V}(r)$, this potential in the new shifted energy scale, such that the ground state of $\hat{V}(r)$ is at zero energy, is

$$\hat{V}_1(r) = \hat{V}(r) - \hat{E}_0. \tag{21}$$

Since the structure of $\hat{V}_1(r)$ is the same as that of $V_1(r)$, we have a similar ansatz for its superpotential

$$\hat{W}(r) = -\frac{\hat{A}}{r} - \hat{B}r + \frac{\hat{D}}{r^3}, \qquad \hat{B} < 0, \hat{D} < 0, \tag{22}$$

having new parameters $\hat{A}$, $\hat{B}$ and $\hat{D}$, which are given by equations similar to (6)–(9) and finally in terms of the original parameters by

$$\hat{B} = -\sqrt{a}, \qquad \hat{D} = -\sqrt{c}, \tag{23}$$

$$\hat{A} = \left(9 + \frac{b}{\sqrt{c}}\right)\Big/2 \tag{24}$$

and

$$\hat{E}_0 = \sqrt{a}\left(12 + \frac{b}{\sqrt{c}}\right). \tag{25}$$

As in equation (6), we have a constraint condition

$$\hat{A}^2 - 2\hat{B}\hat{D} - \hat{A} = l'(l'+1). \tag{26}$$

Using equations (17), (23) and (24), this constraint condition can be expressed in terms of the original parameters as

$$(8\sqrt{c} + b)^2 = 4c\left[l(l+1) + 3 + \frac{b}{\sqrt{c}}\right] + c(1 + 8\sqrt{ac}). \tag{27}$$

In the original energy scale, $\hat{E}_0$ is the ground state energy of $\hat{V}(r) = V_2(r)$ and by SSQM, it is the energy of the first excited state of V_{eff}. This result agrees with equation (14) of [5].

The wavefunction of the first excited state is given by [8]

$$\Psi_1(r) = A^\dagger(r)\Psi_0(r, a, b', c) \tag{28}$$

where

$$A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}r} + W(r). \tag{29}$$

From $W(r)$ and $\Psi_0(r)$ given by equations (2) and (15) (with an appropriate change of parameters in Ψ_0 as indicated in equation (28)), we have

$$\begin{aligned} \Psi_1(r) &= N_1[E + Fr^2 + Gr^{-2}]r^{(7+b\sqrt{c})/2}\mathrm{e}^{[-(\sqrt{a}/2)r^2 - (\sqrt{c}/2)r^{-2}]} \\ E &= -\left(6 + b/\sqrt{c}\right) \\ F &= 2\sqrt{a} \\ G &= -2\sqrt{c} \end{aligned} \tag{30}$$

where N_1 is a normalization constant. We notice that with the application of SSQM in two steps, namely, the introduction of superpotentials for $V_1(r)$ and $\hat{V}_1(r)$, we end up with algebraic expressions for energies of the ground and first excited states; but in addition we are left with two constraints—one for each step, namely, equations (13) and (27)—to be satisfied by the same set of original parameters. Substituting equation (13) into (27) we get

$$b = -6\sqrt{c}. \tag{31}$$

Thus if only condition (13) is satisfied, the energy of the ground state is given by equation (12); if only condition (27) is satisfied, the energy of the first excited state is given by equation (25). On the other hand, if both equations (13) and (27) are simultaneously satisfied, i.e., b and c are related by equation (31), then the energies of both the ground (equation (12)) and the first

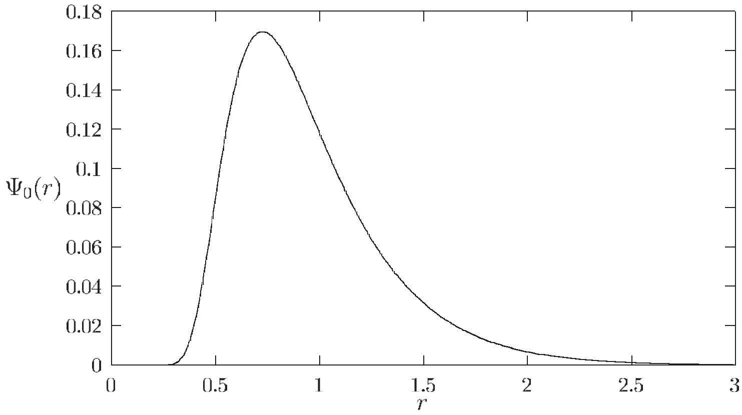

Figure 1. Ground state wavefunction (arbitrary normalization) with $l = 0$, $a = 1$, $c = 3.515\,625$, $b = -11.25$.

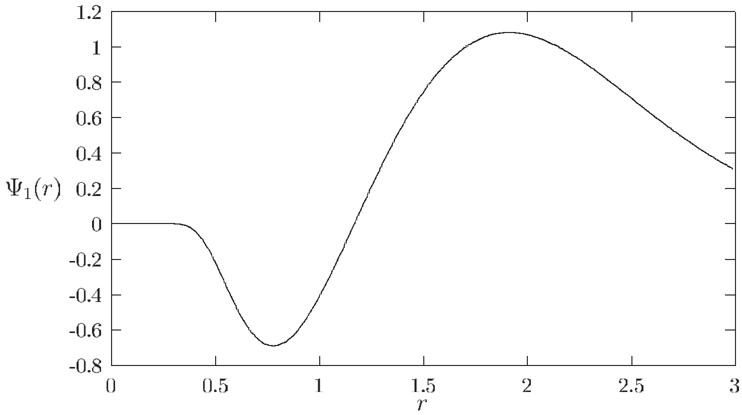

Figure 2. First excited state wavefunction (arbitrary normalization) with $l = 0$, $a = 1$, $c = 3.515\,625$, $b = -11.25$.

excited (equation (25)) states are given algebraically. In this case, substituting equation (31) into (13), we have

$$ac = \left[\frac{16 - (2l + 1)^2}{8}\right]^2. \tag{32}$$

Putting $l = 0, a = 1$, we get $c = (15/8)^2 = 3.515\,625$ and $b = -6\sqrt{c} = -(45/4) = -11.25$. From equation (12) the ground state energy is -2.0, and from equation (25) the energy of the first excited state is 6.0. This is exactly 'solution A' of [7], satisfying equations (12*b*) and (13*b*) of [7]. Varshni pointed out (see the abstract of [7]) that for this case ($b = -11.25$, $c = 3.515\,625$), the constraints for the ground state as well as the first excited state are both satisfied. Equations (31) and (32) relate possible sets of parameters a, b and c for a given l, such that energies of both the ground and the first excited states are given exactly by equations (12) and (25) ($-2\sqrt{a}$ and $6\sqrt{a}$ respectively). The choice of parameters in [5] satisfied condition (13) but not (27), and consequently the algebraic expression for only the ground state energy was given correctly, as later pointed out by Landtman [6] and Varshni [7].

It is clear that in each step of the construction of one more partner of the hierarchy of partner potentials, we need a new constraint condition. From the nature of these constraints (equations (13) and (27)) for a particular choice of a and l, only the parameters b and c can be adjusted to satisfy them simultaneously. Hence, at most two of such sets of constraint conditions can be satisfied simultaneously and at most two of the energy levels can be obtained algebraically, for a severely restricted set of parameters. In figures 1 and 2 we plot the ground and the first excited state wavefunctions (with arbitrary normalization) for the parameters $l = 0, a = 1, b = -11.25, c = 3.515\,625$.

We conclude by remarking that the potential given by equation (1) is conditionally shape invariant in one or at most two steps, for a restricted set of parameters satisfying one or at most two conditions of constraints respectively. In such cases, the energy of one single level or at most two levels can be given in closed algebraic expressions exactly. Thus the potential given by equation (1) is conditionally exactly solvable for one or at most two levels for a selective set of parameters.

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